d'où $\sum_{k=0}^{n} C_n^k e^{i\alpha k} = 2^n \cos^n(\frac{\alpha}{2}) e^{i\frac{n\alpha}{2}}$

3°/ On a : $\sum_{k=0}^{n} C_n^k e^{i\alpha k} = 2^n \cos^n(\frac{\alpha}{2}) e^{i\frac{n\alpha}{2}}$

$\Rightarrow \sum_{k=0}^{n} C_n^k \cos(\alpha k) + i \sum_{k=0}^{n} C_n^k \sin(\alpha k)$

$= 2^n \cos^n(\frac{\alpha}{2}) \cos(\frac{n\alpha}{2}) + i 2^n \cos^n(\frac{\alpha}{2}) \sin(\frac{n\alpha}{2})$

d'où : $\sum_{k=0}^{n} C_n^k \cos(\alpha k) = 2^n \cos^n(\frac{\alpha}{2}) \cos(\frac{n\alpha}{2})$

$\sum_{k=0}^{n} C_n^k \sin(\alpha k) = 2^n \cos^n(\frac{\alpha}{2}) \sin(\frac{n\alpha}{2})$

Solution 13 :

1°/ On sait que pour tout $z \in \mathbf{C} \setminus \{1\}$;

$\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z}$ donc pour $z = e^{i\frac{\pi}{n}} \in \mathbf{C} \setminus \{1\}$, on a :

$$\sum_{k=0}^{n-1} e^{i\frac{k\pi}{n}} = \sum_{k=0}^{n-1} \left(e^{i\frac{\pi}{n}}\right)^k = \frac{1-e^{i\pi}}{1-e^{i\frac{\pi}{n}}} = \frac{1-\cos\pi}{1-e^{i\frac{\pi}{n}}} = \frac{2}{1-e^{i\frac{\pi}{n}}}$$

Conclusion : Pour tout $n \in \mathbf{N}^*$; $\sum_{k=0}^{n-1} e^{i\frac{k\pi}{n}} = \frac{2}{1-e^{i\frac{\pi}{n}}}$

2°/ On a : $\sum_{k=0}^{n-1} e^{i\frac{k\pi}{n}} = \frac{2}{1-e^{i\frac{\pi}{n}}}$

* $\sum_{k=0}^{n-1} e^{i\frac{k\pi}{n}} = \sum_{k=0}^{n-1} \cos(\frac{k\pi}{n}) + i \sum_{k=0}^{n-1} \sin(\frac{k\pi}{n})$ donc $\sum_{k=0}^{n-1} \sin(\frac{k\pi}{n}) = \Im_m(\frac{2}{1-e^{i\frac{\pi}{n}}})$

* $\frac{2}{1-e^{i\frac{\pi}{n}}} = \frac{2}{e^{i\frac{\pi}{2n}}(e^{-i\frac{\pi}{2n}} - e^{i\frac{\pi}{2n}})} = \frac{2}{e^{i\frac{\pi}{2n}}(-2i\sin(\frac{\pi}{2n}))}$